doi:10.1006/jabr.1999.8398, available online at http://www.idealibrary.com on IDEAL®

On the Number of Homomorphisms from a Finite Group to a General Linear Group

Naoki Chigira and Yugen Takegahara

Muroran Institute of Technology, 27-1 Mizumoto, Muroran 050-8585, Japan
E-mail: chigira@mmm.muroran-it.ac.jp, yugen@mmm.muroran-it.ac.jp

and

Tomoyuki Yoshida

Department of Mathematics, Hokkaido University, Kita-10, Nishi-8, Sapporo 060-0810, Japan
E-mail: yoshidat@math.sci.hokudai.ac.jp

Communicated by Walter Feit

Received August 10, 1999

We study the number of homomorphisms from a finite group to a general linear group over a finite field. In particular, we give a generating function of such numbers. Then the Rogers–Ramanujan identities are applicable.

Key Words: general linear group; generating function; q-differentiation; Rogers–Ramanujan identities.

1. INTRODUCTION

Throughout this paper, let G be a finite group and let F be the finite field of q-elements. Let $\mathrm{Hom}(G, GL(n, q))$ denote the set of all homomorphisms from G to the general linear group $GL(n, q)$ of degree n over F. For any simple left FG-module V, the set $\mathrm{End}_{FG}(V)$ consisting of all FG-homomorphisms from V to itself is a division ring by Schur's lemma (see [2, (3.17)]), and hence $\mathrm{End}_{FG}(V)$ is an extension field of F. Suppose that $|G|$ is not a multiple of the characteristic of F, i.e., $|G| \neq 0$ in F. Then, by Maschke's theorem (see [2, (3.14)]), the group ring FG is semisimple. Let $\{V_1, V_2, \ldots, V_r\}$ be a set of representatives of the isomorphism classes of

0021-8693/00 $35.00

simple left FG-modules. For each integer i with $1 \le i \le r$, let d_i be the dimension of V_i over F, and let e_i be the dimension of $\mathrm{End}_{FG}(V_i)$ over F; so $\mathrm{End}_{FG}(V_i)$ is the finite field of q^{e_i}-elements. In Section 2, using the representation theory of finite groups, we will prove [7, p. 27, Theorem]

$$1+\sum_{n=1}^{\infty}\frac{|\mathrm{Hom}(G, GL(n,q))|}{|GL(n,q)|}x^n=\prod_{i=1}^{r}\left(1+\sum_{n=1}^{\infty}\frac{x^{d_i n}}{|GL(n,q^{e_i})|}\right).$$

If G is cyclic, the formula above concerns the number of elements in some conjugacy classes of $GL(n,q)$ (see [4, pp. 270–272]). If $q-1$ is a multiple of $\exp(G)$, then $\mathrm{End}_{FG}(V_i)=F$ for each i, i.e., F is a splitting field for G (see [2, (17.1)]). Also, using the results in [6], we will give the generating functions of the numbers of homomorphisms from a finite cyclic group whose order divides $q-1$ to classical groups over F (see Theorem 3.2).

Since $|GL(n,q)| = (q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$, the Rogers–Ramanujan identities (see [3, (8.10.14),(8.10.15)]) yield

$$1+\sum_{n=1}^{\infty}\frac{1}{|GL(n,q)|}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{-(5n+1)})(1-q^{-(5n+4)})} \tag{RR1}$$

and

$$1+\sum_{n=1}^{\infty}\frac{1}{|GL(n,q)|q^n}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{-(5n+2)})(1-q^{-(5n+3)})}. \tag{RR2}$$

Combining these formulas with the generating function above, we have the following theorem, which includes [7, p. 27, Corollary].

THEOREM 1.1. *Suppose that* $q-1$ *is a multiple of* $\exp(G)$. *Then* r *is the number of conjugacy classes of* G, *and*

$$1+\sum_{n=1}^{\infty}\frac{|\mathrm{Hom}(G, GL(n,q))|}{|GL(n,q)|}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{-(5n+1)})^r(1-q^{-(5n+4)})^r}.$$

If G *is abelian, then*

$$1+\sum_{n=1}^{\infty}\frac{|\mathrm{Hom}(G, GL(n,q))|}{|GL(n,q)|q^n}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{-(5n+2)})^r(1-q^{-(5n+3)})^r}.$$

If $q-1$ is a multiple of $|G/G'|$, where G/G' is the factor group of G by the commutator subgroup G', then $|\mathrm{Hom}(G, GL(n,q))|$ is a multiple of $|G/G'|$ (see [7, p. 28, Theorem]); we can get this fact by applying q-differentiation to the generating function of $|\mathrm{Hom}(G, GL(n,q))|$ (see Section 5). Also, if $q-1$ is a multiple of $\exp(G/G')$, then $|\mathrm{Hom}(G, GL(n,q))|$ is a multiple of $|G/G'|$ (see Theorem 6.1). The conjecture below is correct in such a case.

Conjecture [1]. For finite groups A and G,

$$|\mathrm{Hom}(A, G)| \equiv 0 \bmod \gcd(|A/A'|, |G|).$$

The following theorem was proved in [8]:

THEOREM A. *For a finite group G and a finite abelian group A,*

$$|\mathrm{Hom}(A, G)| \equiv 0 \bmod \gcd(|A|, |G|).$$

If G is not abelian, let d be the smallest d_i with $d_i > e_i$. Using Theorem A, we will prove in Section 4 that

$$\big|\mathrm{Hom}(G, GL(n, q))\big| \equiv 0 \bmod \gcd\big(|G/G'|, (q-1)^{n-[n/d]}\big),$$

where $[n/d]$ denotes the greatest integer not exceeding n/d.

Let $\overline{\mathrm{Hom}}(G, GL(n, q))$ denote the set of homomorphisms from G to $GL(n, q)$ that are not of the block diagonal form

$$\begin{pmatrix} \varphi_1(g) & 0 \\ 0 & \varphi_2(g) \end{pmatrix}$$

for all $g \in G$. In Section 6, we will give the formulas

$$\left(1 + \sum_{n=1}^{\infty} \big|\mathrm{Hom}(G, GL(n, q))\big| x^n\right)\left(1 - \sum_{n=1}^{\infty} \big|\overline{\mathrm{Hom}}(G, GL(n, q))\big| x^n\right) = 1$$

and

$$\big|\overline{\mathrm{Hom}}(G, GL(n, q))\big| = \sum_{k_1+\cdots+k_j=n} (-1)^{j-1} \prod_{i=1}^{j} \big|\mathrm{Hom}(G, GL(k_i, q))\big|,$$

where the summation runs over all sequences $(k_1, k_2, \ldots, k_j)$ of positive integers summing to n.

2. THE FORMULA OF $|\mathrm{Hom}(G, GL(n, q))|$

To start with, we prove the following.

THEOREM 2.1. *Let G be a finite group and let F be the finite field of q-elements. Suppose that $|G|$ is not a multiple of the characteristic of F. Let $\{V_1, V_2, \ldots, V_r\}$ be a set of representatives of the isomorphism classes of simple left FG-modules. For each integer i with $1 \le i \le r$, let d_i be the dimension of V_i over F and let e_i be the dimension of* $\mathrm{End}_{FG}(V_i)$ *over F. Then*

$$\big|\mathrm{Hom}(G, GL(n, q))\big| = \sum_{d_1 n_1+\cdots+d_r n_r=n} \frac{|GL(n, q)|}{\prod_{i=1}^{r} |GL(n_i, q^{e_i})|},$$

where the summation runs over all sequences $(n_1, n_2, \ldots, n_r)$ *of nonnegative integers that satisfy* $d_1 n_1 + d_2 n_2 + \cdots + d_r n_r = n$, *and* $GL(0, q) = \{1\}$. *Consequently,*

$$1 + \sum_{n=1}^{\infty} \frac{\left|\mathrm{Hom}(G, GL(n, q))\right|}{|GL(n, q)|} x^n = \prod_{i=1}^{r} \left(1 + \sum_{n=1}^{\infty} \frac{x^{d_i n}}{|GL(n, q^{e_i})|} \right).$$

Proof. Let V be the n-dimensional vector space over F and let $GL(V)$ be the set of all invertible linear transformations from V to itself. Then $GL(V)$ is a group isomorphic to $GL(n, q)$. Take $\varphi \in \mathrm{Hom}(G, GL(V))$. Then V is a left FG-module with respect to φ; we denote this left FG-module by V_φ. By Maschke's theorem, V_φ is the direct sum of simple left FG-submodules. Suppose that

$$V_\varphi \simeq \bigoplus_{i=1}^{r} V_i^{(n_i)},$$

where $n_1, n_2, \ldots, n_r$ are nonnegative integers that satisfy $d_1 n_1 + d_2 n_2 + \cdots + d_r n_r = n$, and $V_i^{(n_i)}$ denotes the direct sum of n_i copies of V_i for each i. In this decomposition, the sequence $(n_1, n_2, \ldots, n_r)$ is uniquely determined by φ in the sense of the Krull–Schmidt–Azumaya theorem (see [2, (6.12)]). Let $\mathrm{H}(n_1, \ldots, n_r)$ denote the set consisting of all homomorphisms from G to $GL(V)$ that determine the sequence $(n_1, n_2, \ldots, n_r)$ as above. For each $\sigma \in GL(V)$, define a homomorphism ${}^\sigma\varphi$ from G to $GL(V)$ by

$${}^\sigma\varphi(g) = \sigma\varphi(g)\sigma^{-1}$$

for all $g \in G$. Then $\sigma(\in GL(V))$ induces a FG-isomorphism $V_\varphi \simeq V_{{}^\sigma\varphi}$, and hence ${}^\sigma\varphi \in \mathrm{H}(n_1, \ldots, n_r)$. Let $\mathrm{Aut}_{FG}(V_\varphi)$ be the subgroup of $GL(V)$ consisting of all units in $\mathrm{End}_{FG}(V_\varphi)$ and let $GL(V)/\mathrm{Aut}_{FG}(V_\varphi)$ be the set of all left cosets of $\mathrm{Aut}_{FG}(V_\varphi)$ in $GL(V)$. Then there exists a bijection

$$GL(V)/\mathrm{Aut}_{FG}(V_\varphi) \ni \sigma\mathrm{Aut}_{FG}(V_\varphi) \qquad \leftrightarrow \qquad {}^\sigma\varphi \in \mathrm{H}(n_1, \ldots, n_r)$$

for all left cosets $\sigma\mathrm{Aut}_{FG}(V_\varphi)$. Thus we have

$$\left|\mathrm{Hom}(G, GL(n, q))\right| = \sum_{d_1 n_1 + \cdots + d_r n_r = n} \frac{|GL(n, q)|}{\left|\mathrm{Aut}_{FG}\left(\bigoplus_{i=1}^{r} V_i^{(n_i)}\right)\right|}.$$

It follows from Schur's lemma and [2, p. 462, Lemma] that

$$\mathrm{End}_{FG}\left(\bigoplus_{i=1}^{r} V_i^{(n_i)} \right) \simeq \bigoplus_{i=1}^{r} M_{n_i}(\mathrm{End}_{FG}(V_i)),$$

where $M_{n_i}(\mathrm{End}_{FG}(V_i))$ is the ring of $n_i \times n_i$ matrices over $\mathrm{End}_{FG}(V_i)$. Therefore

$$\left|\mathrm{Aut}_{FG}\left(\bigoplus_{i=1}^{r} V_i^{(n_i)}\right)\right| = \prod_{i=1}^{r}\left|GL(n_i, q^{e_i})\right|,$$

because $\mathrm{End}_{FG}(V_i)$ is the field of q^{e_i}-elements for each i. This completes the proof of Theorem 2.1. ∎

The next theorem also follows from the Rogers–Ramanujan identities.

THEOREM 2.2. *Under the hypotheses of Theorem* 2.1,

$$1+\sum_{n=1}^{\infty}\frac{\left|\mathrm{Hom}(G, GL(n,q))\right|}{|GL(n,q)|} = \prod_{n=0}^{\infty}\prod_{i=1}^{r}\frac{1}{(1-q^{-e_i(5n+1)})(1-q^{-e_i(5n+4)})},$$

and if G is abelian, *then*

$$1+\sum_{n=1}^{\infty}\frac{\left|\mathrm{Hom}(G, GL(n,q))\right|}{|GL(n,q)|q^n} = \prod_{n=0}^{\infty}\prod_{i=1}^{r}\frac{1}{(1-q^{-e_i(5n+2)})(1-q^{-e_i(5n+3)})}.$$

Proof. By replacing q with q^{e_i} in the formulas (RR1) and (RR2), we can make the generating function in Theorem 2.1 into the desired formulas. Note that if G is abelian, $d_i = e_i$ for each i (see the proof of Proposition 4.2). ∎

We denote the characteristic of F by p, i.e., q is a power of p. Let C be a finite cyclic group, let P be its Sylow p-subgroup, and let $\{V_1, V_2, \ldots, V_r\}$ be a set of representatives of the isomorphism classes of simple left $F(C/P)$-modules. For each integer i with $1 \le i \le r$, let d_i denote the dimension of V_i over F and put $q_i = |\mathrm{End}_{F(C/P)}(V_i)|(= q^{d_i})$. Let $\mathcal{D}_{|P|}(n)$ denote the set of all partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $\lambda_1 \ge \lambda_2 \ge \cdots \ge 0$, of n with largest part $\lambda_1 \le |P|$. For $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{D}_{|P|}(n)$, put $m_j(\lambda) = \sharp\{s \mid \lambda_s = j\}$ and $\bar{\lambda} = \sum (s-1)\lambda_s$. We have the following.

THEOREM 2.3. *Under the notation above*,

$$1+\sum_{n=1}^{\infty}\frac{\left|\mathrm{Hom}(C, GL(n,q))\right|}{|GL(n,q)|}x^n$$
$$= \prod_{i=1}^{r}\left(1+\sum_{n=1}^{\infty}\sum_{\lambda\in\mathcal{D}_{|P|}(n)}\frac{x^{d_i n}}{q_i^{n+2\bar{\lambda}}}\prod_{j=1}^{|P|}\frac{q_i^{m_j(\lambda)^2}}{\left|GL(m_j(\lambda), q_i)\right|}\right).$$

Proof. It follows from [4, pp. 181–182, 270–272] that

$$\left|\mathrm{Hom}(C, GL(n,q))\right| = \sum_{d_1 n_1 + \cdots + d_r n_r = n} \sum_{i=1}^{r} \sum_{\lambda \in \mathcal{D}_{|P|}(n_i)} \frac{|GL(n,q)|}{q_i^{n_i + 2\bar{\lambda}} \prod_{j=1}^{|P|} \varphi_{m_j(\lambda)}(q_i^{-1})},$$

where

$$\varphi_{m_j(\lambda)}(q_i^{-1}) = (1 - q_i^{-1}) \cdots (1 - q_i^{-m_j(\lambda)}) = \frac{(q_i - 1) \cdots (q_i^{m_j(\lambda)} - 1)}{q_i^{m_j(\lambda)(m_j(\lambda)+1)/2}}.$$

Then

$$\frac{\left|\mathrm{Hom}(C, GL(n,q))\right|}{|GL(n,q)|}$$
$$= \sum_{d_1 n_1 + \cdots + d_r n_r = n} \sum_{i=1}^{r} \sum_{\lambda \in \mathcal{D}_{|P|}(n_i)} \frac{1}{q_i^{n_i + 2\bar{\lambda}}} \prod_{j=1}^{|P|} \frac{q_i^{m_j(\lambda)^2}}{|GL(m_j(\lambda), q_i)|},$$

and so the theorem follows. ∎

3. THE CLASSICAL GROUPS

Let $SL(n,q)$ denote the special linear group over the finite field of q-elements. Let C_2 be a cyclic group of order 2. By the proof of Theorem 2.1, if q is odd, then

$$\left|\mathrm{Hom}(C_2, SL(n,q))\right| = \sum_{k=0}^{[n/2]} \frac{|GL(n,q)|}{|GL(2k,q)||GL(n-2k,q)|}.$$

Hence we have the following.

THEOREM 3.1. *If q is odd, then*

$$1 + \sum_{n=1}^{\infty} \frac{\left|\mathrm{Hom}(C_2, SL(n,q))\right|}{|GL(n,q)|} x^n$$
$$= \left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{|GL(2n,q)|}\right)\left(1 + \sum_{n=1}^{\infty} \frac{x^n}{|GL(n,q)|}\right).$$

Next, let us study the number of homomorphisms from a finite cyclic group to the unitary groups $U(n,q)$, the symplectic groups $Sp(2n,q)$, and the orthogonal groups $O(2n+1,q)$, $O_{+1}(2n,q)$, and $O_{-1}(2n,q)$; the last two orthogonal groups correspond to non-degenerate quadratic forms of index n and $n-1$, respectively. For the orders of these classical groups, see, e.g., [6, pp. 33–34, 57–58].

THEOREM 3.2. *Let C be a finite cyclic group whose order divides $q-1$.*

(1) *Let $q = r^2$, $k = \sharp\{c \in C | c^{r+1} = 1\}$, and $l = (|C| - k)/2$. Then*

$$1 + \sum_{n=1}^{\infty} \frac{|\mathrm{Hom}(C, U(n,q))|}{|U(n,q)|} x^n = \left(1 + \sum_{n=1}^{\infty} \frac{x^n}{|U(n,q)|}\right)^k \left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{|GL(n,q)|}\right)^l.$$

(2) *Suppose that q is odd and that $|C|$ is even. Then*

$$1 + \sum_{n=1}^{\infty} \frac{|\mathrm{Hom}(C, Sp(2n,q))|}{|Sp(2n,q)|} x^n = \left(1 + \sum_{n=1}^{\infty} \frac{x^n}{|Sp(2n,q)|}\right)^2 \left(1 + \sum_{n=1}^{\infty} \frac{x^n}{|GL(n,q)|}\right)^{(|C|-2)/2}.$$

(3) *Suppose that q and $|C|$ are odd. Then*

$$1 + \sum_{n=1}^{\infty} \frac{|\mathrm{Hom}(C, Sp(2n,q))|}{|Sp(2n,q)|} x^n = \left(1 + \sum_{n=1}^{\infty} \frac{x^n}{|Sp(2n,q)|}\right) \left(1 + \sum_{n=1}^{\infty} \frac{x^n}{|GL(n,q)|}\right)^{(|C|-1)/2}.$$

(4) *Suppose that q is even. Let G_n denote $Sp(2n,q) \simeq O(2n+1,q)$, $O_{+1}(2n,q)$, or $O_{-1}(2n,q)$. Then*

$$1 + \sum_{n=1}^{\infty} \frac{|\mathrm{Hom}(C, G_n)|}{|G_n|} x^n = \left(1 + \sum_{n=1}^{\infty} \frac{x^n}{|G_n|}\right) \left(1 + \sum_{n=1}^{\infty} \frac{x^n}{|GL(n,q)|}\right)^{(|C|-1)/2}.$$

Proof. Let us regard C as a subgroup of the cyclic group $F - \{0\}$. For (1), C is a disjoint union of the sets Ω_+, Ω_-, and Ω_0 that satisfy the following conditions:

$$\Omega_0 = \{c \in C | c^{r+1} = 1\},$$
$$\Omega_+ \ni c \quad \Leftrightarrow \quad c^{-r} \in \Omega_-.$$

Then $k = \sharp\Omega_0$ and $l = \sharp\Omega_+ = \sharp\Omega_-$. By [6, pp. 34–36], we have

$$\left|\mathrm{Hom}(C, U(n,q))\right| = \sum_{n_1+\cdots+n_k+2n_{k+1}+\cdots+2n_{k+l}=n} \frac{|U(n,q)|}{\left(\prod_{i=1}^{k} |U(n_i,q)|\right)\left(\prod_{i=k+1}^{k+l} |GL(n_i,q)|\right)},$$

which yields the desired formula. Thus (1) follows.

Let us prove (2). Put $m = 2 + (|C| - 2)/2$. It follows from [6, pp. 36–38] that

$$\left|\mathrm{Hom}(C, Sp(2n,q))\right| = \sum_{n_1+n_2+\cdots+n_m=n} \frac{|Sp(2n,q)|}{|Sp(2n_1,q)|\,|Sp(2n_2,q)|\prod_{i=3}^{m} |GL(n_i,q)|}.$$

Hence we get the desired generating function. Thus (2) follows.

Likewise, (3) and (4) follow from [6, pp. 36–38, 58–62], respectively. ∎

4. SOME PROPERTIES OF $|\mathrm{Hom}(G, GL(n,q))|$

Let us turn to the various formulas of $|\mathrm{Hom}(G, GL(n,q))|$. For a nonnegative integer n, define

$$[n;q] = \begin{cases} 1 + q + \cdots + q^{n-1} & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}$$

and

$$[n;q]! = \begin{cases} [n;q][n-1;q]\cdots[1;q] & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

For a sequence $(n_1, n_2, \ldots, n_r)$ of nonnegative integers summing to n,

$$\begin{bmatrix} n \\ n_1, \ldots, n_r \end{bmatrix}_q = \frac{[n;q]!}{[n_1;q]![n_2;q]!\cdots[n_r;q]!}$$

is called the q-multinomial coefficient. It is well known that the q-multinomial coefficient is a polynomial in q with nonnegative coefficients (see, e.g., [5, 1.3.17]).

PROPOSITION 4.1. *Let G be a finite group and let F be the finite field of q-elements. Suppose that $|G|$ is not a multiple of the characteristic of F. Then $|\mathrm{Hom}(G, GL(n,q))|$ is a polynomial in q with integer coefficients. If $q-1$ is a multiple of $\exp(G)$ and if G is abelian, then $|\mathrm{Hom}(G, GL(n,q))|$ is a polynomial in q with nonnegative integer coefficients.*

Proof. We use the notation of Theorem 2.1. Let $(n_1, n_2, \ldots, n_r)$ be a sequence of nonnegative integers such that $d_1 n_1 + d_2 n_2 + \cdots + d_r n_r = n$. Then

$$\frac{|GL(n,q)|}{\prod_{i=1}^r \left|GL(n_i, q^{e_i})\right|} = \frac{(q-1)^n}{(q-1)^{\sum_{i=1}^r n_i}} \frac{q^{n(n-1)/2}}{q^{\sum_{i=1}^r \frac{e_i n_i (n_i - 1)}{2}}} \frac{[n;q]!}{\prod_{i=1}^r \prod_{j=1}^{n_i} [e_i j; q]},$$

and

$$\frac{[n;q]!}{\prod_{i=1}^r \prod_{j=1}^{n_i} [e_i j; q]} = \begin{bmatrix} n \\ d_1 n_1, \ldots, d_r n_r \end{bmatrix}_q \prod_{i=1}^r \frac{[d_i n_i; q]!}{\prod_{j=1}^{n_i} [e_i j; q]},$$

which is a polynomial in q with nonnegative integer coefficients. Hence the proposition follows from Theorem 2.1. ∎

EXAMPLE 4.1. Let G be a finite abelian group of order r. Suppose that $q-1$ is a multiple of $\exp(G)$. Then

$$|\mathrm{Hom}(G, GL(n,q))| = \sum_{n_1 + \cdots + n_r = n} q^{(n^2 - (n_1^2 + \cdots + n_r^2))/2} \begin{bmatrix} n \\ n_1, \ldots, n_r \end{bmatrix}_q.$$

In particular,

$$\left|\mathrm{Hom}(G, GL(2,q))\right| = r + \frac{r(r-1)}{2}(q + q^2)$$

and

$$\begin{aligned} \left|\mathrm{Hom}(G, GL(3,q))\right| = r + r(r-1)q^2 + \frac{r(r-1)(r+4)}{6} q^3 \\ + \frac{r(r-1)(r+1)}{3} q^4 \\ + \frac{r(r-1)(r-2)}{3} q^5 + \frac{r(r-1)(r-2)}{6} q^6. \end{aligned}$$

PROPOSITION 4.2. *Under the hypotheses of Theorem* 2.1, *if $d_i = e_i$ for each i with $1 \le i \le l$ and if $d_i > e_i$ for each i with $i > l$, then*

$$\begin{aligned} \left|\mathrm{Hom}(G, GL(n,q))\right| = \sum_{k=0}^n \left|\mathrm{Hom}(G/G', GL(k,q))\right| \\ \times \sum_{d_{l+1} n_{l+1} + \cdots + d_r n_r = n-k} \frac{|GL(n,q)|}{|GL(k,q)| \prod_{i=l+1}^r |GL(n_i, q^{e_i})|}. \end{aligned}$$

Proof. Put $F_i = \mathrm{End}_{FG}(V_i)$ for each i. Then

$$FG \simeq \bigoplus_{i=1}^r \mathrm{End}_{F_i}(V_i) \simeq \bigoplus_{i=1}^r M_{d_i/e_i}(F_i)$$

(see [2, (3.22),(3.28),(3.32)]). So every simple left FG-module V with $V = \operatorname{End}_{FG}(V)$ is also a simple left $F(G/G')$-module, and this correspondence gives a bijection between the isomorphism classes of such simple left FG-modules and the isomorphism classes of simple left $F(G/G')$-modules. Thus the proposition follows from Theorem 2.1. ∎

EXAMPLE 4.2. Let S_3 denote the symmetric group on three letters. Suppose that $q-1$ is a multiple of 6. There are three simple left FS_3-modules whose dimensions are 1, 1, 2. Then

$$\begin{aligned}\left|\operatorname{Hom}(S_3, GL(2,q))\right| &= \left|\operatorname{Hom}(C_2, GL(2,q))\right| + q(q^2-1)\\ &= 2 + q + q^2 + q^3 - q\\ &= 2 + q^2 + q^3,\end{aligned}$$

and

$$\begin{aligned}\left|\operatorname{Hom}(S_3, GL(3,q))\right| &= \left|\operatorname{Hom}(C_2, GL(3,q))\right| + 2q^3(q^3-1)(q+1)\\ &= 2\left(1 + q^2 + q^3 + q^4\right) + 2\left(-q^3 - q^4 + q^6 + q^7\right)\\ &= 2\left(1 + q^2 + q^6 + q^7\right).\end{aligned}$$

PROPOSITION 4.3. *Under the hypotheses of Theorem* 2.1, *if d is the smallest d_i with $d_i > e_i$, then*

$$\left|\operatorname{Hom}(G, GL(n,q))\right| \equiv 0 \bmod \gcd\left(|G/G'|, (q-1)^{n-[n/d]}\right).$$

Proof. Let k be an integer with $0 \le k \le n$. By Theorem A, $|\operatorname{Hom}(G/G', GL(k,q))|$ is a multiple of $\gcd(|G/G'|, |GL(k,q)|)$. Also, in Proposition 4.2,

$$\begin{aligned}&\frac{|GL(n,q)|}{|GL(k,q)|\prod_{i=l+1}^{r}|GL(n_i,q^{e_i})|}\\ &\quad= \frac{(q-1)^{n-k}}{(q-1)^{\sum_{i=l+1}^{r} n_i}}\,\frac{q^{(n(n-1)-k(k-1))/2}}{q^{\sum_{i=l+1}^{r}\frac{e_i n_i(n_i-1)}{2}}}\,\frac{[n;q]!}{[k;q]!\prod_{i=l+1}^{r}\prod_{j=1}^{n_i}[e_i j;q]}\\ &\quad= \frac{(q-1)^{n-k}}{(q-1)^{\sum_{i=l+1}^{r} n_i}}\,\frac{q^{(n(n-1)-k(k-1))/2}}{q^{\sum_{i=l+1}^{r}\frac{e_i n_i(n_i-1)}{2}}}\\ &\qquad\times \begin{bmatrix} n \\ k, d_{l+1}n_{l+1}, \ldots, d_r n_r\end{bmatrix}_q \prod_{i=l+1}^{r}\frac{[d_i n_i;q]!}{\prod_{j=1}^{n_i}[e_i j;q]}.\end{aligned}$$

Since $n_{l+1}, n_{l+2}, \ldots, n_r \ge 0$ and $\sum_{i=l+1}^{r} d_i n_i = n-k$, $n-k \ge d\sum_{i=l+1}^{r} n_i$, and hence $n-k-\sum_{i=l+1}^{r} n_i \ge n-k-[(n-k)/d]$. Therefore

$$\left|\operatorname{Hom}(G/G', GL(k,q))\right|\frac{|GL(n,q)|}{|GL(k,q)|\prod_{i=l+1}^{r}|GL(n_i,q^{e_i})|}$$

is a multiple of $\gcd(|G/G'|, (q-1)^{n-[n/d]})$ and so is $|\mathrm{Hom}(G, GL(n, q))|$. This completes the proof of Proposition 4.3. ∎

5. q-DIFFERENTIATION

We will apply q-differentiation (see, e.g., [3, p. 22]) to the series in Theorem 2.1. For a function $u(x)$, define

$$\hat{B}u(x) = \frac{u(x) - u(qx)}{(1-q)x}$$

(q-differentiation), and $\hat{Q}u(x) = u(qx)$. Let $w(x) = u(x)v(x)$. Then

$$\hat{B}w(x) = \hat{B}u(x)v(x) + \hat{Q}u(x)\hat{B}v(x). \tag{D1}$$

For nonnegative integers k and t with $k \geq t$, let $([k], t)$ denote the set consisting of all t-element subsets of $[k] = \{1, 2, \ldots, k\}$ if $t \geq 1$, and let $([k], 0) = \{[0]\}$, where $[0]$ is the empty set. For each $\sigma \in ([k], t)$, put $\hat{T}^{\sigma}u(x) = \hat{T}_k\hat{T}_{k-1}\cdots\hat{T}_1u(x)$, where $\hat{T}_j = \hat{B}$ if $j \in \sigma$, and is $\hat{Q}$ otherwise. Let $\hat{B}^{(k)}u(x) = \hat{T}^{[k]}u(x)$ where $[k] \in ([k], k)$, and let $\hat{Q}^{(k)}u(x) = \hat{T}^{[0]}u(x)$ where $[0] \in ([k], 0)$. Then the formula (D1) yields

$$\hat{B}^{(k)}w(x) = \sum_{t=0}^{k} \sum_{\sigma\in([k],\, t)} \hat{T}^{\sigma}u(x)\hat{B}^{(k-t)}v(x). \tag{D2}$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} k \\ t \end{bmatrix}_q = \begin{bmatrix} k \\ t,\, k-t \end{bmatrix}_q = \frac{[k;q]!}{[t;q]![k-t;q]!}.$$

Then every q-multinomial coefficient is expressed as a product of q-binomial coefficients. We show the following well known result.

LEMMA 5.1. *Under the notation above,*

$$\hat{B}^{(k)}w(x) = \sum_{t=0}^{k} \begin{bmatrix} k \\ t \end{bmatrix}_q \hat{Q}^{(k-t)}\hat{B}^{(t)}u(x)\hat{B}^{(k-t)}v(x).$$

If $g(x) = g_1(x)g_2(x)\cdots g_r(x)$, *then*

$$\hat{B}^{(k)}g(x) = \sum_{k_1+\cdots+k_r=k} \begin{bmatrix} k \\ k_1, \ldots, k_r \end{bmatrix}_q \prod_{i=1}^{r} \hat{Q}^{(\sum_{j=i+1}^{r} k_j)}\hat{B}^{(k_i)}g_i(x).$$

Proof. It suffices to show the first statement. By the definition,

$$\hat{Q}\hat{B}u(x) = \frac{u(qx) - u(q^2x)}{(1-q)qx} = q^{-1}\hat{B}\hat{Q}u(x).$$

Hence, for any $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_t\} \in ([k], t)$ with $\sigma_1 < \sigma_2 < \cdots < \sigma_t$,

$$\begin{aligned}
\hat{T}^{\sigma}u(x) &= q^{\sigma_1 - 1}\hat{T}_k\hat{T}_{k-1}\cdots\hat{T}_{\sigma_2}\hat{Q}^{(\sigma_2-2)}\hat{B}u(x) \\
&= q^{\sum_{s=1}^{j}\sigma_s - \frac{j(j+1)}{2}}\hat{T}_k\hat{T}_{k-1}\cdots\hat{T}_{\sigma_{j+1}}\hat{Q}^{(\sigma_{j+1}-j-1)}\hat{B}^{(j)}u(x) \\
&= q^{\sum_{s=1}^{t}\sigma_s - \frac{t(t+1)}{2}}\hat{Q}^{(k-t)}\hat{B}^{(t)}u(x).
\end{aligned}$$

For each $\sigma \in ([k], t)$, put $\bar{\sigma} = \sum_{s=1}^{t}\sigma_s - t(t+1)/2$ if $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_t\} \neq [0]$, and put $\overline{[0]} = 0$. Then the fact above yields

$$\hat{T}^{\sigma}u(x) = q^{\bar{\sigma}}\hat{Q}^{(k-t)}\hat{B}^{(t)}u(x)$$

for any $\sigma \in ([k], t)$. Thus, by the formula (D2),

$$\hat{B}^{(k)}w(x) = \sum_{t=0}^{k} f_{(k,t)}(q)\hat{Q}^{(k-t)}\hat{B}^{(t)}u(x)\hat{B}^{(k-t)}v(x), \tag{D3}$$

where

$$f_{(k,\,t)}(q) = \sum_{\sigma\in([k],\,t)} q^{\bar{\sigma}}.$$

Let t_0 be an integer with $0 \leq t_0 \leq k$. Put $u(x) = x^{t_0}$, $v(x) = x^{k-t_0}$, and $w(x) = x^k$. Clearly, $\hat{B}^{(k)}w(x) = [k;q]!$. On the other hand, the formula above yields $\hat{B}^{(k)}w(x) = f_{(k,t_0)}(q)[t_0;q]![k-t_0;q]!$. Therefore

$$f_{(k,t_0)}(q) = \frac{[k;q]!}{[t_0;q]![k-t_0;q]!} = \begin{bmatrix} k \\ t_0 \end{bmatrix}_q. \tag{D4}$$

Now, the lemma follows from the formulas (D3) and (D4). ∎

Remark. The formula (D4) is similar to [5, 1.3.19].

The following theorem is an effect of q-differentiation.

THEOREM 5.1. *Under the hypotheses of Theorem* 2.1,

$$\begin{aligned}
&\frac{|\mathrm{Hom}(G, GL(n,q))|}{|GL(n,q)|}\;\frac{[n;q]!}{[n-k;q]!} \\
&\quad = \sum_{d_1n_1+\cdots+d_rn_r=n}\;\sum_{\substack{k_1+\cdots+k_r=k \\ 0\leq k_i\leq d_in_i}} \begin{bmatrix} k \\ k_1, \ldots, k_r \end{bmatrix}_q \\
&\qquad \times \prod_{i=1}^{r} \frac{q^{(d_in_i-k_i)\sum_{j=i+1}^{r}k_j}}{|GL(n_i, q^{e_i})|}\;\frac{[d_in_i;q]!}{[d_in_i-k_i;q]!}
\end{aligned}$$

for each positive integer k with $k \leq n$.

Proof. Let k be a positive integer and let $(k_1, k_2, \ldots, k_r)$ be a sequence of nonnegative integers summing to k. The power series

$$G_q(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{|GL(n,q)|} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{q^{n(n-1)/2}(q-1)^n[n;q]!}$$

converges for all finite values of x because $|q| > 1$. Put $g_i(x) = G_{q^{e_i}}(x^{d_i})$ for each i. Then

$$\begin{aligned}\hat{Q}^{(\sum_{j=i+1}^r k_j)}\hat{B}^{(k_i)} g_i(x) &= \sum_{n \geq k_i/d_i} \frac{q^{(d_i n - k_i)\sum_{j=i+1}^r k_j}}{|GL(n, q^{e_i})|} \\ &\quad \times \frac{1-q^{d_i n}}{1-q} \cdots \frac{1-q^{d_i n - k_i + 1}}{1-q} x^{d_i n - k_i} \\ &= \sum_{n \geq k_i/d_i} \frac{q^{(d_i n - k_i)\sum_{j=i+1}^r k_j}}{|GL(n, q^{e_i})|} \frac{[d_i n; q]!}{[d_i n - k_i; q]!} x^{d_i n - k_i}.\end{aligned}$$

Put $g(x) = g_1(x) g_2(x) \cdots g_r(x)$. Using Lemma 5.1, we have

$$\begin{aligned}\hat{B}^{(k)} g(x) &= \sum_{k_1 + \cdots + k_r = k} \begin{bmatrix} k \\ k_1, \ldots, k_r \end{bmatrix}_q \\ &\quad \times \prod_{i=1}^r \sum_{n \geq k_i/d_i} \frac{q^{(d_i n - k_i)\sum_{j=i+1}^r k_j}}{|GL(n, q^{e_i})|} \frac{[d_i n; q]!}{[d_i n - k_i; q]!} x^{d_i n - k_i}.\end{aligned}$$

On the other hand,

$$\hat{B}^{(k)} g(x) = \sum_{n=k}^{\infty} \frac{|\mathrm{Hom}(G, GL(n,q))|}{|GL(n,q)|} \frac{[n;q]!}{[n-k;q]!} x^{n-k},$$

because

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{|\mathrm{Hom}(G, GL(n,q))|}{|GL(n,q)|} x^n$$

by Theorem 2.1. The theorem follows from these results. ∎

Remark. In this theorem, if $k = n$, the assertion is quite Theorem 2.1.

COROLLARY 5.1. *Let G be a finite abelian group of order r. Suppose that $q-1$ is a multiple of* $\exp(G)$. *Then*

$$\begin{aligned}&|\mathrm{Hom}(G, GL(n,q))| \\ &\quad = \sum_{n_1 + \cdots + n_r = n-1} \frac{|GL(n-1,q)|}{\prod_{i=1}^r |GL(n_i, q)|} \left\{ r + (q-1) \sum_{j=1}^r \Big[n + \sum_{i=1}^{j-1} n_i - n_j - 1; q \Big] \right\}.\end{aligned}$$

Proof. It follows from Theorem 5.1 with $k = 1$ that

$$\begin{aligned}
&\frac{\left|\mathrm{Hom}(G, GL(n,q))\right|}{|GL(n,q)|}[n;q] \\
&\quad = \sum_{n_1+\cdots+n_r=n} \frac{1}{\prod_{i=1}^{r} |GL(n_i,q)|} \sum_{j=1}^{r} q^{\sum_{i=1}^{j-1} n_i}[n_j;q] \\
&\quad = \sum_{n_1+\cdots+n_r=n} \sum_{\substack{j=1 \\ n_j\neq 0}}^{r} \frac{1}{\prod_{i\neq j} q^{n_i(n_i-1)/2}(q-1)^{n_i}[n_i;q]!} \\
&\quad\quad \times \frac{q^{\sum_{i=1}^{j-1} n_i-(n_j-1)}}{q^{(n_j-1)(n_j-2)/2}(q-1)^{n_j}[n_j-1;q]!} \\
&\quad = \sum_{n_1+\cdots+n_r=n} \sum_{\substack{j=1 \\ n_j\neq 0}}^{r} \frac{1}{\prod_{i\neq j} |GL(n_i,q)|} \cdot \frac{q^{\sum_{i=1}^{j-1} n_i-n_j+1}}{|GL(n_j-1,q)|(q-1)}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\left|\mathrm{Hom}(G, GL(n,q))\right| \\
&\quad = \sum_{n_1+\cdots+n_r=n} \sum_{\substack{j=1 \\ n_j\neq 0}}^{r} \frac{|GL(n-1,q)|}{\prod_{i\neq j} |GL(n_i,q)|} \cdot \frac{q^{n+\sum_{i=1}^{j-1} n_i-n_j}}{|GL(n_j-1,q)|} \\
&\quad = \sum_{j=1}^{r} \sum_{n_1+\cdots+n_r=n-1} \frac{|GL(n-1,q)|}{\prod_{i\neq j} |GL(n_i,q)|} \cdot \frac{q^{n+\sum_{i=1}^{j-1} n_i-(n_j+1)}}{|GL(n_j,q)|} \\
&\quad = \sum_{n_1+\cdots+n_r=n-1} \frac{|GL(n-1,q)|}{\prod_{i=1}^{r} |GL(n_i,q)|} \sum_{j=1}^{r} q^{n+\sum_{i=1}^{j-1} n_i-n_j-1}.
\end{aligned}$$

The corollary follows from this formula. ∎

Corollary 5.1, together with Proposition 4.2, implies that if $q-1$ is a multiple of $|G/G'|$, then $|\mathrm{Hom}(G, GL(n,q))|$ is a multiple of $|G/G'|$.

6. FURTHER RESULTS

Let G be a finite group and let F be the finite field of q-elements. Hereafter, we need not suppose $|G| \neq 0$ in F. Let V be an n-dimensional vector space over F and let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. Define $\overline{\mathrm{Hom}}(G, GL(V))$ to be the set of homomorphisms φ from G to $GL(V)$

such that, for each integer k with $1 \le k \le n-1$, the two subspaces V_k and W_k generated by $v_1, v_2, \ldots, v_k$ and $v_{k+1}, v_{k+2}, \ldots, v_n$, respectively, are not both G-submodules of V with respect to φ; i.e.,

$$\overline{\mathrm{Hom}}(G, GL(V)) = \mathrm{Hom}(G, GL(V)) - \bigcup_{k=1}^{n-1} \mathrm{Hom}(G, GL(V_k) \times GL(W_k)).$$

To simplify the notation, put

$$a_n(G; q) = \left|\mathrm{Hom}(G, GL(V))\right|,$$
$$b_n(G; q) = \left|\overline{\mathrm{Hom}}(G, GL(V))\right|.$$

Let us give the following elementary fact.

THEOREM 6.1. *Let $c_n(G;q)$ denote either $a_n(G;q)$ or $b_n(G;q)$. Then $c_n(G;q)$ is a multiple of* $|\mathrm{Hom}(G/G', F^*)|$, *where $F^* = F - \{0\}$. Especially, if $q-1$ is a multiple of* $\exp(G/G')$, *then $c_n(G;q)$ is a multiple of $|G/G'|$.*

Proof. Let $A = G/G'$ and $\hat{A} = \mathrm{Hom}(A, F^*)$. Then $\hat{A}$ is a finite abelian group under the multiplication defined by $(f_1 f_2)(\bar{g}) = f_1(\bar{g}) f_2(\bar{g})$ for all f_1, $f_2 \in \hat{A}$ and $\bar{g} \in A$. There is an isomorphism ρ from $\hat{A}$ into the symmetric group $\Sigma(\mathrm{Hom}(G, GL(V)))$ on $\mathrm{Hom}(G, GL(V))$ defined by

$$\rho(f)(\varphi)(g) = f(gG')\varphi(g)$$

for all $f \in \hat{A}$, $\varphi \in \mathrm{Hom}(G, GL(V))$, and $g \in G$, where $gG' (\in A)$ denotes a coset of G'. So $\hat{A}$ acts on $\mathrm{Hom}(G, GL(V))$ via the action ρ, and this action is semi-regular. Hence $a_n(G;q)$ is a multiple of $|\hat{A}|$. Likewise, $b_n(G;q)$ is a multiple of $|\hat{A}|$. If $q-1$ is a multiple of $\exp(A)$, then $\hat{A}$ is isomorphic to A, and $c_n(G;q)$ is a multiple of $|A|$. This proves the theorem. ∎

We study the relations between $a_n(G;q)$ and $b_n(G;q)$. For a nonnegative integer n, $\mathscr{P}(n)$ denotes the set consisting of all sequences of positive integers summing to n. (A bijection between $\mathscr{P}(n)$ and the set of all subsets of $[n-1]$ is given in [5, p. 14].)

PROPOSITION 6.1. *Under the notation above,*

$$a_n(G;q) = \sum_{j=1}^{n} \sum_{(k_1, \ldots, k_j) \in \mathscr{P}(n)} b_{k_1}(G;q) b_{k_2}(G;q) \cdots b_{k_j}(G;q).$$

Proof. We use induction on n. If $n = 1$, then clearly, $a_1(G;q) = b_1(G;q)$. Assume that $n > 1$. By the definition,

$$a_n(G;q) = \sharp \bigcup_{k=1}^{n-1} \mathrm{Hom}(G, GL(V_k) \times GL(W_k)) + b_n(G;q),$$

and

$$\sharp \bigcup_{k=1}^{n-1} \mathrm{Hom}(G, GL(V_k) \times GL(W_k)) = \sum_{k=1}^{n-1} b_k(G;q) \big|\mathrm{Hom}(G, GL(W_k))\big|.$$

Hence

$$a_n(G;q) = \sum_{k=1}^{n-1} b_k(G;q) a_{n-k}(G;q) + b_n(G;q).$$

By the inductive assumption, we conclude that

$$\begin{aligned} a_n(G;q) &= \sum_{k=1}^{n-1} b_k(G;q) \sum_{j=1}^{n-k} \sum_{(k_1,\ldots,k_j)\in\mathcal{P}(n-k)} b_{k_1}(G;q) b_{k_2}(G;q)\cdots b_{k_j}(G;q) \\ &\quad + b_n(G;q) \\ &= \sum_{j=2}^{n} \sum_{(k_1,\ldots,k_j)\in\mathcal{P}(n)} b_{k_1}(G;q) b_{k_2}(G;q)\cdots b_{k_j}(G;q) + b_n(G;q). \end{aligned}$$

This proves the proposition. ∎

COROLLARY 6.1. *We have*

$$\left(1 + \sum_{n=1}^{\infty} a_n(G;q) x^n\right)\left(1 - \sum_{n=1}^{\infty} b_n(G;q) x^n\right) = 1.$$

Proof. In the proof of Proposition 6.1, we got

$$a_n(G;q) = \sum_{k=1}^{n-1} b_k(G;q) a_{n-k}(G;q) + b_n(G;q).$$

The corollary follows from this fact. ∎

Let us make $\mathcal{P}(n)$ into a partially ordered set (poset) by defining

$$(k_1, k_2, \ldots, k_j) \le (m_1, m_2, \ldots, m_l)$$

if there exists j_i such that $k_1 + k_2 + \cdots + k_{j_i} = m_1 + m_2 + \cdots + m_i$ for each i. (The poset $\mathcal{P}(n)$ is isomorphic to the boolean algebra of rank $n-1$; see [5, p. 107].) Define the mappings f and g from $\mathcal{P}(n)$ to the set of natural numbers by

$$f(\alpha) = a_{k_1}(G;q) a_{k_2}(G;q)\cdots a_{k_j}(G;q)$$

and

$$g(\alpha) = b_{k_1}(G;q) b_{k_2}(G;q)\cdots b_{k_j}(G;q)$$

for $\alpha = (k_1, k_2, \ldots, k_j) \in \mathcal{P}(n)$.

COROLLARY 6.2. *Under the notation above,*

$$f(\beta) = \sum_{\alpha \leq \beta} g(\alpha)$$

for all $\beta \in \mathcal{P}(n)$.

Proof. Let $\beta = (m_1, m_2, \ldots, m_l) \in \mathcal{P}(n)$. Then, by Proposition 6.1,

$$\begin{aligned} f(\beta) &= \prod_{i=1}^{l} a_{m_i}(G; q) \\ &= \prod_{i=1}^{l} \sum_{j=1}^{m_i} \sum_{(k_1, \ldots, k_j) \in \mathcal{P}(m_i)} b_{k_1}(G; q) b_{k_2}(G; q) \cdots b_{k_j}(G; q). \end{aligned}$$

Hence the corollary follows. ∎

For $\alpha = (k_1, k_2, \ldots, k_j)$ and $\beta = (m_1, m_2, \ldots, m_l) \in \mathcal{P}(n)$, put $\ell(\alpha, \beta) = j - l$ if $\alpha \leq \beta$. (The number $\ell(\alpha, \beta)$ is just the length of interval $[\alpha, \beta]$ defined in [5, p. 99].) The Möbius function μ of $\mathcal{P}(n)$ is defined inductively by

$$\begin{aligned} \mu(\alpha, \alpha) &= 1 && \text{for all } \alpha \in \mathcal{P}(n), \\ \mu(\alpha, \beta) &= - \sum_{\alpha \leq \gamma < \beta} \mu(\alpha, \gamma) && \text{for all } \alpha < \beta \text{ in } \mathcal{P}(n). \end{aligned}$$

We have $\mu(\alpha, \beta) = (-1)^{\ell(\alpha, \beta)}$ for all $\alpha \leq \beta$ in $\mathcal{P}(n)$, because

$$\sum_{\alpha \leq \gamma \leq \beta} (-1)^{\ell(\alpha, \gamma)} = \sum_{i=0}^{\ell(\alpha, \beta)} (-1)^i \frac{\ell(\alpha, \beta)!}{i!(\ell(\alpha, \beta) - i)!} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Now the Möbius inversion formula (see [5, 3.7.1]), together with Corollary 6.2, yields the following.

PROPOSITION 6.2. *Under the notation above,*

$$g(\beta) = \sum_{\alpha \leq \beta} (-1)^{\ell(\alpha, \beta)} f(\alpha)$$

for all $\beta \in \mathcal{P}(n)$.

COROLLARY 6.3. *We have*

$$b_n(G; q) = \sum_{j=1}^{n} (-1)^{j-1} \sum_{(k_1, \ldots, k_j) \in \mathcal{P}(n)} a_{k_1}(G; q) a_{k_2}(G; q) \cdots a_{k_j}(G; q).$$

EXAMPLE 6.1. Suppose that G is a finite abelian group of order r and that $q-1$ is a multiple of $\exp(G)$. Then $b_1(G;q) = a_1(G;q) = r$,

$$b_2(G;q) = -r(r-1) + \frac{r(r-1)}{2}(q+q^2),$$

and

$$\begin{aligned} b_3(G;q) &= r(r-1)^2 - r^2(r-1)q - r(r-1)^2q^2 \\ &\quad + \frac{r(r-1)(r+4)}{6}q^3 + \frac{r(r-1)(r+1)}{3}q^4 \\ &\quad + \frac{r(r-1)(r-2)}{3}q^5 + \frac{r(r-1)(r-2)}{6}q^6. \end{aligned}$$

EXAMPLE 6.2. Suppose that $q-1$ is a multiple of 6. Then $b_1(S_3;q) = a_1(S_3;q) = 2$,

$$b_2(S_3;q) = -2 + q^2 + q^3,$$

and

$$b_3(S_3;q) = 2(1 - q^2 - 2q^3 + q^6 + q^7).$$

Combining Corollary 6.3 with Proposition 4.3, we have the following.

PROPOSITION 6.3. *Under the hypotheses of Proposition* 4.3,

$$b_n(G;q) \equiv 0 \bmod \gcd\big(|G/G'|, (q-1)^{n-[n/d]}\big).$$

Proof. For any $(k_1, k_2, \ldots, k_j) \in \mathcal{P}(n)$,

$$a_{k_1}(G;q)a_{k_2}(G;q)\cdots a_{k_j}(G;q)$$

is a multiple of $\gcd(|G/G'|, (q-1)^{\sum_{i=1}^{j}(k_i-[k_i/d])})$ by Proposition 4.3, and furthermore, $\sum_{i=1}^{j}(k_i - [k_i/d]) \geq n - [n/d]$. Hence the proposition follows from Corollary 6.3. ∎

REFERENCES

1. T. Asai and T. Yoshida, $|\mathrm{Hom}(A, G)|$, II, *J. Algebra* **160** (1993), 273–285.
2. C. W. Curtis and I. Reiner, "Methods of Representation Theory," Vol. I, Wiley–Interscience, New York, 1981.
3. G. Gasper and M. Rahman, "Basic Hypergeometric Series," Encyclopedia of Mathematics and its Applications, Vol. 35 (G.-C. Rota, Ed.), Cambridge Univ. Press, Cambridge, UK, 1990.
4. I. G. Macdonald, "Symmetric Functions and Hall Polynomials," 2nd ed., Oxford Univ. Press, Oxford, 1995.

5. R. P. Stanley, "Enumerative Combinatorics," Vol. I, Cambridge Univ. Press, Cambridge, UK, 1997.
6. G. E. Wall, On the conjugacy classes in the unitary, symplectic and orthogonal groups, *J. Austral. Math. Soc.* **3** (1963), 1–62.
7. T. Yoshida, Groups and generating functions, *in* "Groups and Combinatorics," Proceedings of Symp. RIMS, Vol. **794**, 18–29, 1992.
8. T. Yoshida, $|\mathrm{Hom}(A, G)|$, *J. Algebra* **156** (1993), 125–156.